

INTERNATIONAL A-LEVEL FURTHER MATHEMATICS FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

January 2024

Version: 1.0 Final



2 4 1 X F M 0 3 / M S

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from [oxfordaqa.com](https://www.oxfordaqa.com)

Copyright information

OxfordAQA retains the copyright on all its publications. However, registered schools/colleges for OxfordAQA are permitted to copy material from this booklet for their own internal use, with the following important exception: OxfordAQA cannot give permission to schools/colleges to photocopy any material that is acknowledged to a third party even for internal use within the centre.

Copyright © 2024 OxfordAQA International Examinations and its licensors. All rights reserved.

Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
✓ or ft	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–x EE	Deduct x marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)
ISW	Ignore subsequent working

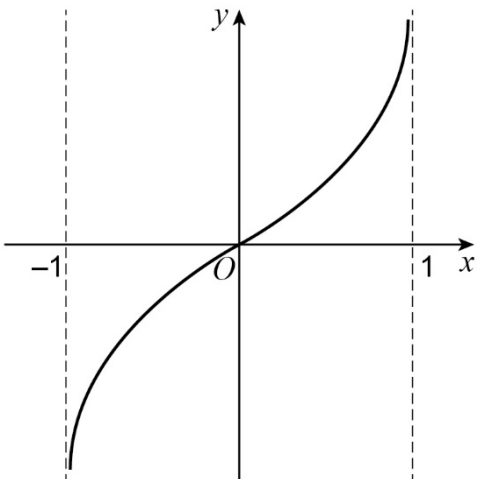
Q	Answer	Marks	Comments
1	$(2t+1, 5t+4, 2t+1.5)$ $2t+1 + 2(5t+4) + 3(2t+1.5) = 18$ $18t = 4.5 \quad t = 0.25$ $A(1.5, 5.25, 2)$	B1 M1 A1	Correct general point on L seen or used PI by a correct linear equation in a single variable Substituting their general point into the equation for Π to obtain an equation in a single variable and obtaining a value for the variable Correct coordinates ACF Do not condone answer given as a vector
	Question 1 Total	3	

Q	Answer	Marks	Comments
2(a)	$\left[\tan \theta = -\frac{1}{\sqrt{3}} \right]$ $\left[-2\sqrt{3} + 2i = \right] 4e^{i\frac{5\pi}{6}}$	B1 B1	B1 for $r = 4$ B1 for $\theta = \frac{5\pi}{6}$
		2	

Q	Answer	Marks	Comments
2(b)	$z^4 = 4e^{i\left(\frac{5\pi}{6} + 2n\pi\right)}$ $z = \sqrt[4]{4} e^{i\left(\frac{5\pi}{24} + \frac{2n\pi}{4}\right)}$ $z = \sqrt{2} e^{i\left(\frac{5\pi}{24}\right)}, \quad z = \sqrt{2} e^{i\left(\frac{17\pi}{24}\right)},$ $z = \sqrt{2} e^{i\left(\frac{29\pi}{24}\right)}, \quad z = \sqrt{2} e^{i\left(\frac{41\pi}{24}\right)}$	B1ft M1 M1 A1	B1ft: $r = \sqrt[4]{4}$ oe ft the fourth root of their value for r in part (a) M1: $z = \sqrt[4]{4} e^{i\left(\frac{5\pi}{24}\right)}$ Divides their θ in part (a) by 4 to get their θ in (b) Obtains at least three of the first four positive correct angles ft $\frac{\text{their } \theta \text{ in (a)}}{4} + \frac{n\pi}{2}$ CAO
		4	

	Question 2 Total	6	
--	------------------	---	--

Q	Answer	Marks	Comments
3(a)(i)	$y = \tanh^{-1} x$ $\frac{dy}{dx} = \frac{1}{1-x^2}$ [When $x = 0$] $\left[\frac{dy}{dx} = \right] 1$	B1	B0 if incorrect $\frac{dy}{dx}$ used
		1	

Q	Answer	Marks	Comments
3(a)(ii)		B1	Correct shaped graph with a positive gradient at the origin and asymptotic behaviour at $x = -1$ and $x = 1$
		1	

Q	Answer	Marks	Comments
3(b)	$\tanh^{-1}\left(\frac{1+x}{2}\right) = \frac{1}{2}\ln\left(\frac{3+x}{1-x}\right)$ $\tanh^{-1}\left(\frac{1-x}{2}\right) = \frac{1}{2}\ln\left(\frac{3-x}{1+x}\right)$ $\tanh^{-1}\left(\frac{1+x}{2}\right) + \tanh^{-1}\left(\frac{1-x}{2}\right)$ $= \frac{1}{2}\ln\left(\frac{(3+x)(3-x)}{(1-x)(1+x)}\right)$ $\frac{3}{2}\ln 3 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln \frac{27}{2}$ $\frac{(9-x^2)}{(1-x^2)} = \frac{27}{2} \Rightarrow 18-2x^2 = 27-27x^2$ $\Rightarrow 25x^2 = 9$ $\Rightarrow x = \pm \frac{3}{5}$	M1 A1 B1 M1 A1	$\tanh^{-1} X = \frac{1}{2}\ln\left(\frac{1+X}{1-X}\right)$ used at least once $\frac{1}{2}\ln\left(\frac{(3+x)(3-x)}{(1-x)(1+x)}\right)$ oe single logarithmic term $\frac{3}{2}\ln 3 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln \frac{27}{2}$ seen or used Elimination of logs to obtain a single quadratic equation in x $[x =] \pm \frac{3}{5}$ oe
		5	

	Question 3 Total	7	
--	-------------------------	----------	--

Q	Answer	Marks	Comments
4	$\frac{r^2 + r + 1}{r(r+1)} = 1 + \frac{1}{r(r+1)}$ $\frac{1}{r(r+1)} = \frac{B}{r} + \frac{C}{r+1} \Rightarrow 1 = B(r+1) + Cr$ $\frac{r^2 + r + 1}{r(r+1)} = 1 + \frac{1}{r} - \frac{1}{r+1}$ $\sum_{r=1}^n \frac{r^2 + r + 1}{r(r+1)} = \sum_{r=1}^n 1 + \sum_{r=1}^n \left(\frac{1}{r} - \frac{1}{r+1} \right)$ $= n + \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots$ $+ \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right)$ $\sum_{r=1}^n \frac{r^2 + r + 1}{r(r+1)} = 1 + n - \frac{1}{n+1}$	B1 M1 A1 B1 M1 A1	$A=1$ or $\frac{r^2 + r + 1}{r(r+1)} = 1 + \frac{1}{r(r+1)}$ PI by either $B=1$ or $C=-1$ $1 + \frac{1}{r} - \frac{1}{r+1}$ $\sum_{r=1}^n 1 = n$ seen or used Method of differences shown with at least the first two and last two terms of $\sum_{r=1}^n \left(\frac{1}{r} - \frac{1}{r+1} \right)$ seen AG Must be convincingly shown, including evidence of cancelling
	Question 4 Total	6	

Q	Answer	Marks	Comments
5	$\cos x \frac{dy}{dx} + y = \cos^2 x + \sin x$ $\Rightarrow \frac{dy}{dx} + y \sec x = \cos x + \tan x$ I.F. is $e^{\int \sec x \, dx}$ I.F. = $e^{\ln(\sec x + \tan x)} = \sec x + \tan x$ $\frac{d}{dx}(y(\sec x + \tan x))$ $= (\cos x + \tan x)(\sec x + \tan x)$ $y(\sec x + \tan x)$ $= \int (\cos x + \tan x)(\sec x + \tan x) \, dx$ $y(\sec x + \tan x)$ $= \int (1 + \tan^2 x + \sin x + \sec x \tan x) \, dx$ $y(\sec x + \tan x) = \tan x - \cos x + \sec x [+A]$ $y = 1 + \frac{A - \cos x}{\sec x + \tan x}$ when $x = \frac{\pi}{3}$, $y = 1 \Rightarrow A = \frac{1}{2}$ $y = 1 + \frac{0.5 - \cos x}{\sec x + \tan x}$	M1 A1 M1 A1 A1ft A1	$e^{\int \sec x \, dx}$ I.F. = $\sec x + \tan x$ or $\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$ Multiplying both sides of the rearranged DE by their I.F. and integrating LHS to get $y \times$ I.F. oe Condone absence of $+A$ $y = f(x)$ with ACF for $f(x)$ eg $y = \sin(x) - \frac{1}{2}\tan(x) + \frac{1}{2}\sec(x)$
	Question 5 Total	6	

Q	Answer	Marks	Comments
6(a)	$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4$	B1	
		1	

Q	Answer	Marks	Comments
6(b)	$e^{\cos(2x)-1} = e^{-2x^2 + \frac{2}{3}x^4} = e^{-2x^2} e^{\frac{2}{3}x^4}$ $= \left(1 + (-2x^2) + \frac{(-2x^2)^2}{2} + \dots \right) \left(1 + \frac{2}{3}x^4 + \dots \right)$ $= (1 - 2x^2 + 2x^4 + \dots) \left(1 + \frac{2}{3}x^4 + \dots \right)$ $= 1 - 2x^2 + \frac{8}{3}x^4 + \dots$	M1 A1	ft their $1 + ax^2 + bx^4$ in part (a) oe eg $e^{\cos(2x)-1} = e^{-2x^2 + \frac{2}{3}x^4}$ $= 1 + \left(-2x^2 + \frac{2}{3}x^4 \right) + \frac{1}{2}(4x^4 + \dots)$ $1 - 2x^2 + \frac{8}{3}x^4$
		2	

Q	Answer	Marks	Comments
6(c)	$e^{\cos(2x)} = e \left(1 - 2x^2 + \frac{8}{3}x^4 + \dots \right)$ $\lim_{x \rightarrow 0} \left(\frac{e - e^{\cos(2x)}}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{e - e \left(1 - 2x^2 + \frac{8}{3}x^4 + \dots \right)}{x^2} \right)$ $= \lim_{x \rightarrow 0} \left(\frac{2ex^2 - \frac{8}{3}ex^4 + \dots}{x^2} \right) = \lim_{x \rightarrow 0} \left(2e - \frac{8}{3}ex^2 + \dots \right)$ $\lim_{x \rightarrow 0} \left(\frac{e - e^{\cos(2x)}}{x^2} \right) = 2e$	B1ft M1 A1	ft their $1 + px^2 + qx^4$ in part (b) Substitutes series expansion and divides numerator and denominator by x^2 to reach the form $\lim_{x \rightarrow 0} [P + O(x^2)]$ A0 if not from $\lim_{x \rightarrow 0} \left(2e - \frac{8}{3}ex^2 \right)$
		3	

	Question 6 Total	6	
--	------------------	---	--

Q	Answer	Marks	Comments
7(a)	The interval of integration is infinite	E1	oe
		1	

Q	Answer	Marks	Comments
7(b)	$[I=] \int \frac{x-3}{e^x} dx = \int (x-3)e^{-x} dx$ $u = x-3, \quad \frac{dv}{dx} = e^{-x}$ $[I=] = -(x-3)e^{-x} - \int -e^{-x} (dx)$ $= -(x-3)e^{-x} - e^{-x} \quad [+c]$ $\int_3^{\infty} \frac{x-3}{e^x} dx = \lim_{a \rightarrow \infty} \int_3^a (x-3)e^{-x} dx$ $= \lim_{a \rightarrow \infty} \left(-(a-3)e^{-a} - e^{-a} \right) - \left(-e^{-3} \right)$ $\left[= \lim_{a \rightarrow \infty} \left(-ae^{-a} + 2e^{-a} \right) - \left(-e^{-3} \right) \right]$ $\lim_{a \rightarrow \infty} \left(a e^{-a} \right) = 0$ $\int_3^{\infty} \frac{x-3}{e^x} dx = e^{-3}$	M1 A1 M1 E1 A1	$\frac{du}{dx} = 1$, $v = \pm e^{-x}$ used in correct integration by parts formula PI by correct integration oe $-xe^{-x} + 2e^{-x}$ Evidence of limit ∞ having been replaced by a (oe) at any stage and $\lim_{a \rightarrow \infty}$ seen or taken at any stage with no remaining lim relating to 3 Must be explicitly stated Accept if stated in the more general format First 3 marks must have been scored but can be awarded even if previous E1 not awarded provided limits clearly substituted and no errors seen
		5	

	Question 7 Total	6	
--	------------------	---	--

Q	Answer	Marks	Comments
8(a)	For $n = 1$, $\sum_{r=1}^n (r^3 + 3r^5) = 1^3 + 3 \times 1^5 = 1 + 3 = 4$ and $\frac{1}{2}n^3(n+1)^3 = \frac{1}{2} \times 1 \times 2^3 = \frac{1}{2} \times 8 = 4$ Assume formula true for $n = k$ (*), so $\sum_{r=1}^{k+1} (r^3 + 3r^5) = \sum_{r=1}^k (r^3 + 3r^5) + (k+1)^3 + 3(k+1)^5$ $= \frac{1}{2}k^3(k+1)^3 + (k+1)^3 + 3(k+1)^3(k+1)^2$ $= \frac{1}{2}(k+1)^3(k^3 + 2 + 6(k+1)^2)$ $= \frac{1}{2}(k+1)^3(k^3 + 6k^2 + 12k + 8)$ $= \frac{1}{2}(k+1)^3(k+2)(k^2 + 4k + 4) = \frac{1}{2}(k+1)^3(k+2)^3$ Hence formula is true for $n = k + 1$ (**) and since true for $n = 1$ (***), formula $\sum_{r=1}^n (r^3 + 3r^5) = \frac{1}{2}n^3(n+1)^3 \text{ is true for}$ $n = 1, 2, 3, \dots$ by induction (****)	B1 M1 A1 A1 E1	Correct values to show formula true for $n = 1$ Must see 4 preceded by a relevant unsimplified numerical expression for both the LHS and the RHS Assumes formula true for $n = k$ and considers $\sum_{r=1}^{k+1} (r^3 + 3r^5)$ Must be convincingly shown Must have (*), (**), (***), present, previous 4 marks scored and a final statement (****) clearly indicating that it relates to positive integers
		5	

Q	Answer	Marks	Comments
8(b)	$\frac{n^2}{4}(n+1)^2 + \sum_{r=1}^n 3r^5 = \frac{1}{2}n^3(n+1)^3$	M1	Splits the LHS of formula and uses the formula for sum of cubes given in the formulae booklet
	$\frac{(3N)^2}{4}(3N+1)^2 + 3\sum_{r=1}^{3N} r^5 = \frac{1}{2}(3N)^3(3N+1)^3$	m1	Replaces n by $3N$ (condone $3n$ for $3N$)
	$\sum_{r=1}^{3N} r^5 = \frac{1}{6}(3N)^3(3N+1)^3 - \frac{(3N)^2}{12}(3N+1)^2$		$\sum_{r=1}^n r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$
	$= N^2(3N+1)^2 \left(\frac{9}{2}N(3N+1) - \frac{3}{4} \right)$ $= \frac{3}{4}N^2(3N+1)^2(6N(3N+1)-1)$ $= \frac{3}{4}N^2(3N+1)^2(18N^2+6N-1)$	A1	may be obtained after the M1 line before replacing n by $3N$ AG Must be convincingly shown Condone if N is written as n
		3	

	Question 8 Total	8	
--	-------------------------	----------	--

Q	Answer	Marks	Comments
9(a)(i)	$\begin{bmatrix} -1 & 4 & k \\ 2 & 3 & 6 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\Rightarrow \begin{bmatrix} -x+4y+kz \\ 2x+3y+6z \\ x+3y-2z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\begin{aligned} -2x+4y+kz &= 0; & x+3y-3z &= 0 \\ 2x+2y+6z &= 0 \Rightarrow x+y+3z &= 0; \end{aligned}$ $\Rightarrow 2x+4y=0 \Rightarrow x=-2y$ $\Rightarrow 8y+kz=0 \quad \text{and} \quad y=3z \Rightarrow k=-24$	M1 A1 A1	Use of $\mathbf{Mv} = \mathbf{v}$ with at least two rows of $\mathbf{Mv}$ correct oe eg $\begin{vmatrix} -2 & 4 & k \\ 2 & 2 & 6 \\ 1 & 3 & -3 \end{vmatrix} = 0$ oe eg correct unsimplified expansion of determinant $k = -24$
		3	

Q	Answer	Marks	Comments
9(a)(ii)	$\frac{x}{-6} = \frac{y}{3} = z$	M1 A1	M1: Either a general point on the line or a direction vector $\lambda \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$ PI A1: Correct Cartesian equations Accept eg $-x = 2y = 6z$
		2	

Q	Answer	Marks	Comments
9(b)	$\det \mathbf{M} = 3k + 64$	B1	Correct expression for $\det \mathbf{M}$
	Cofactor matrix		
	$\begin{bmatrix} -24 & 10 & 3 \\ 3k+8 & 2-k & 7 \\ 24-3k & 2k+6 & -11 \end{bmatrix}$	M1	One complete row or column or diagonal correct
	Inverse matrix $\mathbf{M}^{-1}$	A2	All nine entries correct else A1 for at least six entries correct
	$= \frac{1}{3k+64} \begin{bmatrix} -24 & 3k+8 & 24-3k \\ 10 & 2-k & 2k+6 \\ 3 & 7 & -11 \end{bmatrix}$	M1	Transpose of their cofactors with no more than one further error and division by their $\det \mathbf{M} (\neq 0)$
		A1	Correct $\mathbf{M}^{-1}$ scores 6 marks
		6	

	Question 9 Total	11	
--	-------------------------	-----------	--

Q	Answer	Marks	Comments
10(a)	Aux. equation $m^2 + 2m + 1 = 0$; $m = -1, -1$ $y_{CF} = (Ax + B)e^{-x}$ $y_{PI} = a \sin 4x + b \cos 4x$ $y_{PI}' = 4a \cos 4x - 4b \sin 4x$ $y_{PI}'' = -16a \sin 4x - 16b \cos 4x$ $16a \sin 4x - 16b \cos 4x + 8a \cos 4x - 8b \sin 4x$ $+ a \sin 4x + b \cos 4x = \sin 4x + 38 \cos 4x$ $-15a - 8b = 1$; $-15b + 8a = 38 \Rightarrow a = 1, b = -2$ $y_{PI} = \sin 4x - 2 \cos 4x$ $[y_{GS} =] (Ax + B)e^{-x} + \sin 4x - 2 \cos 4x$	M1 A1 M1 M1 A1 B1ft	Forming and solving the correct aux. equation. PI by correct value(s) of m seen/used Correct CF $y_{PI} = a \sin 4x + b \cos 4x$ seen or used y_{PI}' and y_{PI}'' both of the form $\pm c \sin 4x \pm d \cos 4x$ and substitution into the differential equation Correct particular integral Their CF + their PI with exactly two arbitrary constants
		6	

Q	Answer	Marks	Comments
10(b)	When $x=0$, $y=4 \Rightarrow B-2=4$, $B=6$	M1	$f(0)=4$ or $f'(0)=0$ or $f''(0)=34$ or series expansions used with their GS as far as finding a value for one of the arbitrary constants
	When $x=0$, $\frac{dy}{dx}=0 \Rightarrow A-B+4=0$, $A=2$		
	When $x=0$ $\frac{d^2y}{dx^2}=34 \Rightarrow -2A+B+32=34$, $A=2$	M1	$f(0)=4$ or $f'(0)=0$ or $f''(0)=34$ or series expansions used with their GS as far as finding a value for the remaining arbitrary constant
	$[y=f(x)=] (2x+6)e^{-x} + \sin 4x - 2 \cos 4x$	A1	$(2x+6)e^{-x} + \sin 4x - 2 \cos 4x$ oe PI by the correct $f\left(\frac{\pi}{16}\right)$
	$\left[f\left(\frac{\pi}{16}\right)=\right] \left(\frac{\pi}{8}+6\right)e^{-\frac{\pi}{16}} - \frac{1}{\sqrt{2}}$	A1	ACF but must be exact with no trigonometric functions
		4	

	Question 10 Total	10	
--	-------------------	----	--

Q	Answer	Marks	Comments
11(a)	Direction of normal to Π_1 is $\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \\ 2 \end{bmatrix}$ Magnitude of $\begin{bmatrix} 16 \\ 8 \\ 2 \end{bmatrix} = \sqrt{16^2 + 8^2 + 2^2} = \sqrt{324}$ Direction cosines $\frac{16}{\sqrt{324}}, \frac{8}{\sqrt{324}}, \frac{2}{\sqrt{324}}$ Required sum $= \frac{16}{18} + \frac{8}{18} + \frac{2}{18} = \frac{26}{18} = \frac{13}{9}$	M1 A1 B1ft M1 A1ft	Forming correct cross product. PI by a correct direction vector oe ft their direction vector ft oe $-\frac{13}{9}$ ft one incorrect element in the direction vector for the normal
		5	

Q	Answer	Marks	Comments
11(b)	Direction of L is $\begin{bmatrix} 16 \\ 8 \\ 2 \end{bmatrix} \times \begin{bmatrix} 6 \\ c \\ 2 \end{bmatrix}$ $= \begin{bmatrix} 16 - 2c \\ -20 \\ 16c - 48 \end{bmatrix} = t \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$ $[t = 4 \Rightarrow] c = 4$ Equation of Π_1 is $\mathbf{r} \cdot \begin{bmatrix} 16 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 16 \\ 8 \\ 2 \end{bmatrix} = 38$ $(p, q, 7)$ lies in both planes so $16p + 8q + 14 = 38$ and $6p + qc + 14 = 14$ $\Rightarrow p = 6, q = -9$	M1 M1 A1 B1ft M1 A1	or $\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ c \\ 2 \end{bmatrix} = 0$ or $12 - 5c + 8 = 0$ oe ft their direction vector. If using the given equation for Π_1 , award this mark for $\lambda = -2, \mu = 3$ Substitutes $(p, q, 7)$ into their equations of Π_1 and Π_2 $p = 6$ and $q = -9$
		6	

	Question 11 Total	11	
--	--------------------------	-----------	--

Q	Answer	Marks	Comments
12(a)	$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$ $\cos 5\theta$ $= \cos^5 \theta + 10 \cos^3 \theta (i \sin \theta)^2 + 5 \cos \theta (i \sin \theta)^4$ $= \cos^5 \theta + 10 i^2 \cos^3 \theta (1 - \cos^2 \theta)$ $+ 5 i^4 \cos \theta (1 - \cos^2 \theta)^2$ $= \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta)$ $+ 5 \cos \theta (1 - 2 \cos^2 \theta + \cos^4 \theta)$ $= \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta$ $- 10 \cos^3 \theta + 5 \cos^5 \theta$ $\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$	B1 M1 M1 A1 A1	or $\cos 5\theta = \operatorname{Re}((\cos \theta + i \sin \theta)^5)$ Replacing $\sin^2 \theta$ with $(1 - \cos^2 \theta)$
		5	

Q	Answer	Marks	Comments
12(b)	Roots of $\cos 5\theta = 1$ are $\theta = [0,] \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$ So roots of $16x^5 - 20x^3 + 5x - 1 = 0$ are $[\cos 0 (=1),] \cos \frac{2\pi}{5}, \cos \frac{4\pi}{5}, \cos \frac{6\pi}{5}, \cos \frac{8\pi}{5}$ $16x^5 - 20x^3 + 5x - 1 = 0$ $(x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1) = 0$ The quartic equation whose roots are $\cos \frac{2\pi}{5}, \cos \frac{4\pi}{5}, \cos \frac{6\pi}{5}, \cos \frac{8\pi}{5}$ is $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$	M1 M1 M1 A1	$16x^5 - 20x^3 + 5x - 1$ $= (x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1)$
		4	

Q	Answer	Marks	Comments
12(c)	[Since $\cos(2\pi - \alpha) = \cos \alpha$] $\cos \frac{4\pi}{5} = \cos \frac{6\pi}{5}; \quad \cos \frac{8\pi}{5} = \cos \frac{2\pi}{5}$ For the quartic equation in part (b) Sum of root = $2 \cos \frac{2\pi}{5} + 2 \cos \frac{6\pi}{5} = -\frac{16}{16} = -1$ Product of roots = $\cos^2 \frac{2\pi}{5} \cos^2 \frac{6\pi}{5} = \frac{1}{16}$ Since $\cos \frac{2\pi}{5} > 0$ and $\cos \frac{6\pi}{5} < 0$, $\cos \frac{2\pi}{5} \cos \frac{6\pi}{5} = -\frac{1}{4}$ $x^2 - \left(-\frac{1}{2}\right)x + \left(-\frac{1}{4}\right) = 0$ The quadratic equation with integer coefficients whose roots are $\cos \frac{2\pi}{5}$ and $\cos \frac{6\pi}{5}$ is $4x^2 + 2x - 1 = 0$	M1 m1 E1 A1	Both results seen or used Uses sum of roots $= 2 \cos \frac{2\pi}{5} + 2 \cos \frac{6\pi}{5} = -\frac{16}{16}$ and Product of roots $= \cos^2 \frac{2\pi}{5} \cos^2 \frac{6\pi}{5} = \frac{1}{16}$
		4	

Question 12 Total

13

Q	Answer	Marks	Comments
13(a)(i)	$\frac{dy}{dt} = -(\cosh t)^{-2} \sinh t = -\tanh t \operatorname{sech} t$	M1	Either form, condone sign error
	$\frac{dx}{dt} = \operatorname{sech}^2 t$	A1	Both derivatives correct
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \operatorname{sech}^4 t + \tanh^2 t \operatorname{sech}^2 t$	M1	Squaring and adding their derivatives
	$= \operatorname{sech}^2 t (\operatorname{sech}^2 t + \tanh^2 t) = \operatorname{sech}^2 t$	A1	
	$s = \int_{-1}^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{-1}^1 \operatorname{sech} t dt$	A1	AG Must be convincingly shown
		5	

Q	Answer	Marks	Comments
13(a)(ii)	$s = \int_{-1}^1 \frac{2}{e^t + e^{-t}} dt$	B1	$\operatorname{sech} t = \frac{2}{e^t + e^{-t}}$ or $\operatorname{sech} t = \frac{\cosh t}{1 + \sinh^2 t}$ or PI
	$s = \int_{-1}^1 \frac{2e^t}{e^{2t} + 1} dt = \int_{e^{-1}}^e \frac{2}{u^2 + 1} du$	M1	Valid substitution seen or PI
	$s = \left[2 \tan^{-1} u \right]_{e^{-1}}^e$	M1	$k \tan^{-1} u$ or $k \tan^{-1}(\sinh t)$ oe
	$s = 2 \tan^{-1}(e) - 2 \tan^{-1}(e^{-1})$	A1	oe eg condone $4 \tan^{-1}(e) - \pi$
		4	

Q	Answer	Marks	Comments
13(b)	$y = \operatorname{sech} t$, $\tanh^2 t + \operatorname{sech}^2 t = 1$ $x^2 + y^2 = 1$ [Possible values of x] $-1 < x < 1$ [Possible values of y] $0 < y \leq 1$	 B1 B1 B1	
		3	

	Question 13 Total	12	
--	--------------------------	-----------	--

Q	Answer	Marks	Comments
14(a)	$9(x^2 + y^2) = 36 - 24x + 4x^2$ $9r^2 = (6 - 2x)^2$; $9r^2 = (6 - 2r \cos \theta)^2$ $\pm 3r = (6 - 2r \cos \theta)$ $\frac{6}{r} = 2 \cos \theta \pm 3$; Since $r \geq 0$, take + sign $\frac{6}{r} = 2 \cos \theta + 3 \Rightarrow r = \frac{6}{3 + 2 \cos \theta}$	M1 M1 A1 A1	$x^2 + y^2 = r^2$ or $y = r \sin \theta$ used $x = r \cos \theta$ used Condone missing $\pm$ AG Must be convincingly shown
		4	

Q	Answer	Marks	Comments
14(b)	$5 + 4 \cos \theta = \frac{6}{3 + 2 \cos \theta}$ $8 \cos^2 \theta + 22 \cos \theta + 9 = 0$ $\Rightarrow \cos \theta = -0.5$ [only as $\cos \theta \neq -2.25$] When $\cos \theta = -0.5$, $r = 3$; $P\left(3, \frac{2\pi}{3}\right)$, $Q\left(3, -\frac{2\pi}{3}\right)$, $\angle POQ = \frac{2\pi}{3}$ area of triangle OPQ $= \frac{1}{2}(3)^2 \sin\left(\frac{2\pi}{3}\right) = \frac{9}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{9\sqrt{3}}{4}$	M1 A1 A1 A1	M1 : Forms a quadratic equation in $\cos \theta$ and solves to find a value for $\cos \theta$ A1 : Correct value of $\cos \theta$ Either correct coordinates of P and Q or $r = 3$ and $\angle POQ = \frac{2\pi}{3}$ seen/used AG Must be convincingly shown
		4	

Q	Answer	Marks	Comments
14(c)	Equation of PQ is $r \cos \theta = -1.5$ $(5 + 4 \cos \theta) \cos \theta = -1.5$ $8 \cos^2 \theta + 10 \cos \theta + 3 = 0$ $\cos \theta = -\frac{1}{2}$ [P and Q]; $\cos \theta = -\frac{3}{4}$ [S and T] when $\cos \theta = -\frac{3}{4}$, $r = 2$ coordinates of S and T are $(2, \pi - \alpha)$ and $(2, -\pi + \alpha)$ where $\cos \alpha = \frac{3}{4}$ required area = $0.5(2)(2) \sin 2\alpha - 2(0.5) \int_{\pi-\alpha}^{\pi} (5 + 4 \cos \theta)^2 d\theta$ $= \frac{3\sqrt{7}}{4} - [33\theta + 40 \sin \theta + 4 \sin 2\theta]_{\pi-\alpha}^{\pi}$ $= \frac{3\sqrt{7}}{4} - \{ -(-33\alpha + 40 \sin \alpha - 4 \sin 2\alpha) \}$ $= \frac{3\sqrt{7}}{4} - 33 \cos^{-1} \left(\frac{3}{4} \right) + 40 \frac{\sqrt{7}}{4} - 8 \frac{\sqrt{7}}{4} \frac{3}{4}$ $= \frac{37}{4} \sqrt{7} - 33 \cos^{-1} \left(\frac{3}{4} \right)$	B1ft M1 A1 M1 B1 B1 A1	ft their values for r and $\cos \theta$ Solving their $r \cos \theta = -k$ with $r = 5 + 4 \cos \theta$ to obtain two possible values of either $\cos \theta$ or r Correct polar coordinates of S and T PI by the correct area of SOT oe Representing the required area in any correct mathematical form B1 for correct exact value for area of triangle SOT B1 for correct integration of $(5 + 4 \cos \theta)^2$
		7	

	Question 14 Total	15	
--	-------------------	----	--